Math 208 Midterm 2

Feb 26th, 2024

Name______________________________ **Section number**______

There are 5 problems, which are 35 points in total.

- (Academic honesty) **Sign your name below:**
  " I have not given or received any unauthorized help on this exam"
  [signature] ____________________
- You are allowed to use a summary sheet.
- No other resources are allowed (Internet, graphing calculator, other humans, . . .).
- **All answers must be justified or with necessary steps. You will receive at most** 1 **point for an answer without any explanation.**
- Please write down your initials **in each page**.
- **DO NOT WRITE ON THE BACK OF EACH PAGE.** Instead, use the last page for extra space.

Initials ___________________

Problem 1.(10 points) Let A be the following matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 2 & -2 & 2 & 2 \\ 1 & 0 & 3 & 2 \end{bmatrix} \qquad \to \qquad B = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and it has the echelon form B.

(1)(3 points) Find a basis for row(A) and find its dimension.

(2)(3 points) Find a basis for col(A) and find its dimension.

(3)(1 point) Find rank(A).

(3)(3 point) Find a basis for null(A) and compute nullity(A).

Initials ____________________

Problem 2. (8 points) Consider the following vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Let A be a 3×3 matrix of which the three columns are $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$, that is, $A = [\ \mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3\ ]$.

(1)(2 points) Is $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{b}\}$ a basis for $\mathbb{R}^3$? Briefly explain your answer.

(2)(4 points) Find A^{-1}. Show all calculations.

(3)(2 points) Using A^{-1} find a solution $\mathbf{x}$ to the equation $A\mathbf{x} = \mathbf{b}$.

Initials ___________________

Problem 3. (6 points)

(1) (4 points) If possible, give an example of a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ such that:

(a) T is not one-to-one, and

(b)

$$\text{range}(T) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \, : \, x - y + 2z = 0 \right\}.$$

Explain why your linear transformation has the desired properties. If impossible, explain why.

(2) (2 points) Solve for the matrix X in the equation $(D+XB)^{-1}XA = C$. Assume that all matrices are $n \times n$ and invertible as needed.

Initials ____________________

Problem 4. (7 points)

(1) (3 points) Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ given by $T(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

Consider the subspace $S = \{\mathbf{x} \in \mathbb{R}^3 : T(\mathbf{x}) = \mathbf{x}\}$. Find a basis for S and find $\dim(S)$.

Initials ___________________

(2) (4 points) Find $\det(A)$ using the cofactor expansion and then find $\det(A^{-1})$, where

$$A = \begin{bmatrix} -1 & 1 & 1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Your answer: $\det(A) =$ ______ and $\det(A^{-1}) =$ ______.

Initials ____________________

Problem 5. (4 points) Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation such that

$$T(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad T(\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad T(\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad T(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix}) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Is T one-to-one? Explain your answer.